

Manifolds with positive second H. Weyl curvature invariant

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Abstract

The second H. Weyl curvature invariant of a Riemannian manifold, denoted h_4 , is the second curvature invariant which appears in the well known tube formula of H. Weyl. It coincides with the Gauss-Bonnet integrand in dimension 4. A crucial property of h_4 is that it is nonnegative for Einstein manifolds, hence it provides a geometric obstruction to the existence of Einstein metrics in dimensions ≥ 4 , independently from the sign of the Einstein constant. This motivates our study of the positivity of this invariant. Here in this paper we prove many constructions of metrics with positive second H. Weyl curvature invariant, generalizing similar well known results for the scalar curvature.

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1 Introduction and statement of the results

Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 4$. Let R , cR and c^2R denote respectively the Riemann curvature tensor, Ricci tensor and the scalar curvature of (M, g) . The *second Hermann Weyl curvature invariant*, which throughout this paper shall be written in abridged form as **shwci** and denoted by h_4 , can be defined by

$$h_4 = \|R\|^2 - \|cR\|^2 + \frac{1}{4}\|c^2R\|^2$$

A crucial property of h_4 is that it is nonnegative for Einstein manifolds (see section 3 below), and so it provides a new geometric obstruction to the

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existence of Einstein metrics independently from the sign of the Einstein constant. In particular, the manifolds which do not admit any metric with positive *shwci* cannot admit any Einstein metric.

Recall that in dimensions greater than 4, we do not know any topological restriction for a manifold to be Einstein. If one requires that the Einstein constant to be positive, then one has two geometric obstructions $cR > 0$ and $c^2R > 0$.

It would be then with a great benefit to have a classification of manifolds with positive *shwci*.

Here in this paper, we shall inaugurate the study of the positivity properties of this important invariant.

In section 2, we introduce and study in general the Hermann Weyl curvature invariants which appear in the tube formula. Many examples are included.

In section 3, we study separately the case of the second invariant. We prove that it is nonnegative for Einstein manifolds and nonpositive for conformally flat manifolds with zero scalar curvature. The limit cases are also discussed.

Then we prove theorem A which states that positive (resp. nonnegative) p -curvature implies positive (resp. nonnegative) *shwci* where $p = [(n+1)/2]$. In particular, positive (resp. nonnegative) sectional curvature implies positive (resp. nonnegative) *shwci*. Also if $n \geq 8$, positive (resp. nonnegative) isotropic curvature implies positive (resp. nonnegative) *shwci*.

Then one can apply our previous constructions in the class of manifolds with positive p -curvature (see [5, 6, 7]) to get many examples of metrics with positive *shwci*.

In section 4 we prove the following useful theorem. It generalizes a similar result for the scalar curvature.:

Theorem B. *Suppose that the total space M of a Riemannian submersion is compact and the fibers (with the induced metric) are with positive *shwci* then the manifold M admits a Riemannian metric with positive *shwci*.*

The section is then ended with two applications of this theorem.

In section 5, we prove the following stability theorem in the class of compact manifolds with positive *shwci*.:

Theorem C. *If a manifold M is obtained from a compact manifold X by surgery in codimension ≥ 5 , and X admits a metric of positive *shwci*, then so does M .*

In particular, the connected sum of two compact manifolds of dimensions ≥ 5 and each one is with positive shwci admits a metric with positive shwci.

Theorem C generalizes a celebrated theorem of Gromov-Lawson and Schoen-Yau for the scalar curvature.

As a consequence of the previous theorem we prove that there are no restrictions on the fundamental group of a compact manifold of dimension ≥ 6 to carry a metric with positive shwci.

Finally, let us mention that it would be interesting to prove, like in the case of the scalar curvature, that every manifold with nonnegative shwci that is not identically zero admits a metric with positive shwci.

2 The H. Weyl curvature invariants

Let $\Lambda^*M = \bigoplus_{p \geq 0} \Lambda^p M$ denote the ring of differential forms on M , where M is as above. Considering the tensor product over the ring of smooth functions, we define $\mathcal{D} = \Lambda^*M \otimes \Lambda^*M = \bigoplus_{p,q \geq 0} \mathcal{D}^{p,q}$ where $\mathcal{D}^{p,q} = \Lambda^p M \otimes \Lambda^q M$. It is graded associative ring and called the ring of double forms on M .

The ring of curvature structures on M ([3]) is the ring $\mathcal{C} = \sum_{p \geq 0} \mathcal{C}^p$ where \mathcal{C}^p denotes symmetric elements in $\mathcal{D}^{p,p}$. We denote by \mathcal{C}_1 (resp. \mathcal{C}_2 , \mathcal{C}_0) the subring of curvature structures satisfying the first (resp. the second, both the first and second) Bianchi identity.

The standard inner product and the Hodge star operator $*$ on $\Lambda^p M$ can be extended in a standard way to \mathcal{D} and they satisfy the following properties, see [4] for the proof:

$$g\omega = *c*\omega \quad (1)$$

for all $\omega \in \mathcal{D}$, where c denotes the contraction map. Also for all $\omega_1, \omega_2 \in \mathcal{D}$, we have

$$\langle g\omega_1, \omega_2 \rangle = \langle \omega_1, c\omega_2 \rangle \quad (2)$$

that is the contraction map is the formal adjoint of the multiplication map by the metric g . Furthermore, we have for all $\omega_1, \omega_2 \in \mathcal{D}^{p,q}$

$$\langle \omega_1, \omega_2 \rangle = *(\omega_1 * \omega_2) = *(*\omega_1.\omega_2) \quad (3)$$

and

$$** = (-1)^{(p+q)(n-p-q)} Id \quad (4)$$

Where Id is the identity map on $\mathcal{D}^{p,q}$.

Next, we define the H. Weyl curvature invariants:

Definition. The $2q$ -Hermann Weyl curvature invariant, denoted h_{2q} , is the complete contraction of the tensor R^q , precisely,

$$h_{2q} = \frac{1}{(2q)!} c^{2q} R^q$$

where R^q denotes the multiplication of R with itself q -times in the ring \mathcal{C} .

Remark that $h_2 = \frac{1}{2} c^2 R$ is one half the scalar curvature and if n is even then h_n is (up to a constant) the Gauss-Bonnet integrand.

Note that in [4], it is proved that

$$h_{2q} = * \frac{1}{(n-2q)!} g^{n-2q} R^q \quad (5)$$

2.1 Examples

1. Let (M, g) be with constant sectional curvature λ , then

$$R = \frac{\lambda}{2} g^2 \quad \text{and} \quad R^q = \frac{\lambda^q}{2^q} g^{2q}$$

And therefore h_{2q} is constant and equals to

$$h_{2q} = * \frac{1}{(n-2q)!} g^{n-2q} R^q = * \frac{\lambda^q}{2^q (n-2q)!} g^n = \frac{\lambda^q n!}{2^q (n-2q)!}$$

In particular,

$$h_4 = \frac{n(n-1)(n-2)(n-3)}{4} \lambda^2 \quad (6)$$

2. Let (M, g) be a Riemannian product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) . If we index by i the invariants of the metric g_i for $i = 1, 2$, then

$$R = R_1 + R_2 \quad \text{and} \quad R^q = (R_1 + R_2)^q = \sum_{i=0}^q C_i^q R_1^i R_2^{q-i}$$

consequently, a straightforward calculation shows that

$$\begin{aligned}
h_{2q} &= \frac{c^{2q} R^q}{(2q)!} = \sum_{i=0}^q C_i^q \frac{c^{2q}}{(2q)!} (R_1^i R_2^{q-i}) \\
&= \sum_{i=0}^q C_i^q \frac{c^{2i} R_1^i}{(2i)!} \frac{c^{2q-2i} R_2^{q-i}}{(2q-2i)!} \\
&= \sum_{i=0}^q C_i^q (h_{2i})_1 (h_{2q-2i})_2
\end{aligned}$$

In particular,

$$h_4 = (h_4)_1 + \frac{1}{2} \text{scal}_1 \text{scal}_2 + (h_4)_2 \quad (7)$$

where scal denotes the scalar curvature.

3. Let (M, g) be a hypersurface of the Euclidean space. If B denotes the second fundamental form at a given point, then the Gauss equation shows that

$$R = \frac{1}{2} B^2 \quad \text{and} \quad R^q = \frac{1}{2^q} B^{2q}$$

Consequently, if $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of B , then the eigenvalues of R^q are $\frac{(2q)!}{2^q} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_{2q}}$ where $i_1 < \dots < i_{2q}$. Consequently,

$$h_{2q} = \frac{(2q)!}{2^q} \sum_{1 \leq i_1 < \dots < i_{2q} \leq n} \lambda_{i_1} \dots \lambda_{i_{2q}}$$

So they coincide, up to a constant, with the symmetric functions in the eigenvalues of B .

4. Let (M, g) be a conformally flat manifold. Then it is well known that at each point of M , the Riemann curvature tensor is determined by a symmetric bilinear form h , in the sense that $R = g.h$. Consequently, $R^q = g^q h^q$.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of eigenvectors of h and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of h .

Then it is not difficult to see that all the tensors R^q are also diagonalizable by the $2q$ -vectors $e_{i_1} \wedge \dots \wedge e_{i_{2q}}$, $i_1 < \dots < i_{2q}$. Their eigenvalues are of the form

$$R^q(e_1 \wedge \dots \wedge e_{2q}, e_1 \wedge \dots \wedge e_{2q}) = (q!)^2 \sum_{1 \leq i_1 < \dots < i_q \leq 2q} \lambda_{i_1} \dots \lambda_{i_q}$$

Consequently we get

$$h_{2q} = \frac{(n-q)!q!}{(n-2q)!} \sum_{1 \leq i_1 < \dots < i_q \leq n} \lambda_{i_1} \dots \lambda_{i_q}$$

5. Let $g_t = tg$ for $t > 0$. If we index by t the invariants of g_t then

$$R_t = tR \quad \text{and} \quad R_t^q = t^q R^q$$

and therefore

$$(h_{2q})_t = \frac{1}{t^q} h_{2q} \quad (8)$$

Let us now recall some other useful facts from [4] which shall be used later.

Following Kulkarni we call the elements in $\ker c \subset D^{p,q}$ effective elements of $D^{p,q}$, and shall be denoted by $E^{p,q}$.

Recall the following orthogonal decomposition of $D^{p,q}$:

$$D^{p,q} = E^{p,q} \oplus gE^{p-1,q-1} \oplus g^2E^{p-2,q-2} \oplus \dots \oplus g^r E^{p-r,q-r} \quad (9)$$

where $r = \min\{p, q\}$.

With respect to the previous decomposition, if $\omega = \sum_{i=0}^p g^{p-i} \omega_i \in C_1^p$ and $n = 2p$, then (see [4])

$$*\omega = \sum_{i=0}^p (-1)^i g^{p-i} \omega_i \quad (10)$$

Also let us recall the following lemma from [4]:

Lemma 2.1 *Let $\omega_1 \in E_1^r, \omega_2 \in E_1^s$ be effectives then*

$$\langle g^p \omega_1, g^q \omega_2 \rangle = 0 \quad \text{if} \quad (p \neq q) \quad \text{or} \quad (p = q \quad \text{and} \quad r \neq s)$$

$$\langle g^p \omega_1, g^p \omega_2 \rangle = p! \left(\prod_{i=0}^{p-1} (n - 2r - i) \right) \langle \omega_1, \omega_2 \rangle \quad \text{if} \quad p \geq 1 \quad \text{and} \quad r = s$$

3 The second H. Weyl curvature invariant

With respect to the previous orthogonal decomposition 9, the Riemann curvature tensor decomposes to $R = \omega_2 + g\omega_1 + g^2\omega_0$, where

$$\begin{aligned}\omega_0 &= \frac{1}{2n(n-1)}c^2R \\ \omega_1 &= \frac{1}{n-2}(cR - \frac{1}{n}gc^2R)\end{aligned}$$

and ω_2 is the Weyl tensor, it is defined by the previous decomposition of R .

Corollary 6.5 in [4] shows that

$$h_4 = \frac{1}{(n-4)!}[n!|\omega_0|^2 - (n-2)!|\omega_1|^2 + (n-4)!|\omega_2|^2] \quad (11)$$

using lemma 2.1 we can easily check that

$$\begin{aligned}\|\omega_2\|^2 &= \|R\|^2 - \frac{1}{n-2}\|cR\|^2 + \frac{1}{2(n-1)(n-2)}\|c^2R\|^2 \\ \|\omega_1\|^2 &= \frac{1}{(n-2)^2}(\|cR\|^2 - \frac{1}{n}\|c^2R\|^2) \\ \|\omega_0\|^2 &= \frac{1}{4n^2(n-1)^2}\|c^2R\|^2\end{aligned} \quad (12)$$

and consequently using formula 11 we obtain another useful expression for h_4 as follows:

$$h_4 = \|R\|^2 - \|cR\|^2 + \frac{1}{4}\|c^2R\|^2 \quad (13)$$

The following theorem was first proved in [4].

Theorem 3.1 *Let (M, g) be a Riemannian manifold of dimension ≥ 4 .*

1. *If (M, g) is an Einstein manifold then $h_4 \geq 0$. Furthermore $h_4 \equiv 0$ if and only if (M, g) is flat.*
2. *If (M, g) is conformally flat with zero scalar curvature then $h_4 \leq 0$. Furthermore $h_4 \equiv 0$ if and only if (M, g) is flat.*

Proof. If (M, g) is conformally flat then $\omega_2 = 0$ and then

$$\begin{aligned}h_4 &= \frac{1}{(n-4)!}[n!|\omega_0|^2 - (n-2)!|\omega_1|^2] \\ &= \frac{n-3}{n-2}[\frac{n}{4(n-1)}\|c^2R\|^2 - \|cR\|^2]\end{aligned}$$

From which is clear that if $c^2 R = 0$ then $h_4 \leq 0$ and $h_4 \equiv 0$ if and only if the metric is Ricci flat and hence is flat. This proves the first part of the theorem.

Next, if (M, g) is Einstein then $\omega_1 = 0$ and hence

$$\begin{aligned} h_4 &= \frac{1}{(n-4)!} [n! \|\omega_0\|^2 + (n-4)! \|\omega_2\|^2] \\ &= \|R\|^2 + \frac{n-4}{4n} (c^2 R)^2 \end{aligned}$$

From which it is clear that $h_4 \geq 0$ and $h_4 \equiv 0$ if and only if the metric is flat. This completes the proof of the theorem. \blacksquare

Recall that (see [6, 7]) the p -curvature of (M, g) , denoted s_p for $1 \leq p \leq n-2$, is a function defined on the p -Grassmanian bundle of the manifold. Its value at a tangent p -plane P is the average of the sectional curvatures of all 2-planes orthogonal to P . In particular s_0 is the scalar curvature and s_{n-2} is twice the sectional curvature.

The following theorem provides a relation between the positivity of the p -curvature and the shwci.:

Theorem A. *Let (M, g) be a Riemannian manifold of dimension $n \geq 4$ and with nonnegative (resp. positive) p -curvature such that $p \geq \frac{n}{2}$, then the shwci of (M, g) is nonnegative (resp. positive). Furthermore, it vanishes if and only if the manifold is flat.*

Proof. Suppose $n = 2(k+2)$ is even, $k \geq 0$. Since

$$R = \omega_2 + g\omega_1 + g^2\omega_0$$

then

$$g^k R = g^k \omega_2 + g^{k+1} \omega_1 + g^{k+2} \omega_0$$

and by formula 10 we have

$$*g^k R = g^k \omega_2 - g^{k+1} \omega_1 + g^{k+2} \omega_0$$

On the other hand since $s_{k+2} \geq 0$, then both the tensors $g^k R$ and $*g^k R$ are with positive sectional curvature, hence

$$[g^k \omega_2 + g^{k+2} \omega_0](e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}}) \geq g^{k+1} \omega_1(e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}})$$

and

$$[g^k \omega_2 + g^{k+2} \omega_0](e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}}) \geq -g^{k+1} \omega_1(e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}})$$

for all orthonormal vectors $e_{i_1}, \dots, e_{i_{k+2}}$, and therefore

$$[g^k \omega_2 + g^{k+2} \omega_0](e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}}) \geq |g^{k+1} \omega_1(e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}})|$$

Consequently, using formulas 5 and 3, we get

$$h_4 = * \frac{1}{(n-4)!} g^{n-4} R^2 = * \frac{1}{(2k)!} (g^k R \cdot g^k R) = \frac{1}{(2k)!} \langle g^k R, * g^k R \rangle$$

and hence using lemma 2.1 and considering an orthonormal basis diagonalizing cR , we obtain

$$\begin{aligned} (2k)! h_4 &= \langle g^k \omega_2 + g^{k+2} \omega_0, g^k \omega_2 + g^{k+2} \omega_0 \rangle - \langle g^{k+1} \omega_1, g^{k+1} \omega_1 \rangle \\ &\geq \sum_{i_1 < \dots < i_{k+2}} [(g^k \omega_2 + g^{k+2} \omega_0)(e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}})]^2 - \|g^{k+1} \omega_1\|^2 \\ &\geq \sum_{i_1 < \dots < i_{k+2}} [g^{k+1} \omega_1(e_{i_1}, \dots, e_{i_{k+2}}, e_{i_1}, \dots, e_{i_{k+2}})]^2 - \|g^{k+1} \omega_1\|^2 = 0 \end{aligned}$$

The same proof works for strict inequality. Also it is clear that if $s_{k+2} \geq 0$ and $h_4 \equiv 0$ then $s_{k+2} \equiv 0$ so that the metric is flat.

To complete the proof, note that if the dimension of the manifold $n = 2p + 1 \geq 5$ is odd then one can consider the product $M \times S^1$. It is of even dimension $2(p+1)$ and with nonnegative (resp. positive) $(p+1)$ -curvature therefore by formula 7 we have

$$h_4(M) = h_4(M \times S^1) \geq 0 \text{ (resp. } > 0 \text{)}$$

this completes the proof of the theorem. ■

Corollary 3.2 *1. A Riemannian manifold of dimension ≥ 4 and with nonnegative (resp. positive) sectional curvature is with nonnegative (resp. positive) shwci. Furthermore, $h_4 \equiv 0$ if and only if the metric is flat.*

2. A Riemannian manifold of dimension ≥ 8 and with nonnegative (resp. positive) isotropic curvature is with nonnegative (resp. positive) shwci. Furthermore, $h_4 \equiv 0$ if and only if the metric is flat.

Proof. Straightforward since positive sectional curvature implies positive p -curvature and positive isotropic curvature implies the positivity of the p -curvature for all $p \leq n - 4$, see [5]. ■

Remarks.

1. If the dimension of the manifold is even, say $n = 2q$, Hopf conjecture states that if the sectional curvature is positive then so is the Gauss-Bonnet integrand that is h_{2q} . Then one can ask the more general question:

Does positive sectional curvature implies positive h_{2k} , for all $2 \leq 2k \leq n$?

This is now true for $k = 1, 2$ by the previous theorem, and it remains an open question for $k \geq 3$.

2. Theorem A generalizes a result of Thorpe [9] for the dimension $n = 4$.

It results from the previous corollary that Lie groups with a biinvariant metric and normal homogeneous Riemannian manifolds are with nonnegative shwci. Furthermore using our previous results on the p -curvature [6], [7] and the above theorem we can easily prove the following corollaries:

Corollary 3.3 *1. Let G be a compact connected Lie group with rank r such that $r < \lfloor \frac{\dim G + 1}{2} \rfloor$ endowed with a biinvariant metric b then (G, b) is with positive shwci.*

In particular, if G is simple then it is with positive shwci.

2. *If G/H is a normal homogeneous Riemannian manifold such that the rank r of G satisfies $r < \lfloor \frac{\dim(G/H) + 1}{2} \rfloor$ then it is with positive shwci.*

Corollary 3.4 *If a compact manifold M admits a smooth action of a compact connected simple Lie group with rank r satisfying $r > \lfloor \frac{\dim M + 1}{2} \rfloor$ then it admits a metric with positive shwci.*

4 Proof of theorem B

Let (M, g) and (B, \check{g}) be two Riemannian manifolds, and let $\pi : (M, g) \rightarrow (B, \check{g})$ a Riemannian submersion. We define, for every $t \in \mathbf{R}$, a new Riemannian metric g_t on the manifold M by multiplying the metric g by t^2 in the vertical directions. Recall that $\forall m \in M$, we have a natural orthogonal decomposition of the tangent space at m

$$T_m M = \mathcal{V}_m \oplus \mathcal{H}_m$$

where \mathcal{V}_m is the tangent to the fiber at m and \mathcal{H}_m is the horizontal space, so that

$$\begin{aligned} g_t|_{\mathcal{V}_m} &= t^2 g \\ g_t|_{\mathcal{H}_m} &= g \\ g_t(\mathcal{V}_m, \mathcal{H}_m) &= 0 \end{aligned}$$

Note that in this case, $\pi : (M, g_t) \rightarrow (B, \check{g})$ is still a Riemannian submersion with the same horizontal and vertical distributions (see [1], [6]).

In the following we shall index by t all the invariants of the metric g_t , and in the case $t = 1$ we omit the index 1.

Also, We make under a hat “ $\hat{\cdot}$ ” (resp. under a check “ $\check{\cdot}$ ”) the invariants of the fibers with the induced metric (resp. of the basis B).

Using lemma 2.1 in [6] it is easy to show that for all g_t -unit tangent vectors e_1, e_2, e_3, e_4 we have

$$R_t(e_1, e_2, e_3, e_4) = O\left(\frac{1}{t}\right) \quad \text{if one of these vectors is horizontal}$$

and that

$$R_t(e_1, e_2, e_3, e_4) = \frac{1}{t^2} \hat{R}(te_1, te_2, te_3, te_4) + O(1) \quad \text{if the four vectors are vertical}$$

Consequently, if $\{e_1, e_2, \dots, e_n\}$ is a g_t -orthonormal basis such that $\{e_1, \dots, e_q\} \in \mathcal{V}_m$ and $\{e_{q+1}, \dots, e_n\} \in \mathcal{H}_m$, then

$$\begin{aligned} (\|R_t\|_t)^2 &= \sum_{1 \leq i < j \leq n, 1 \leq k < l \leq n} [R_t(e_i, e_j, e_k, e_l)]^2 \\ &= \frac{1}{t^4} \sum_{1 \leq i < j \leq q, 1 \leq k < l \leq q} [\hat{R}(te_i, te_j, te_k, te_l)]^2 + O\left(\frac{1}{t^2}\right) \\ &= \frac{1}{t^4} \|\hat{R}\|^2 + O\left(\frac{1}{t^2}\right) \\ (\|Ric_t\|_t)^2 &= \sum_{1 \leq i, j \leq n} [Ric_t(e_i, e_j)]^2 \\ &= \frac{1}{t^4} \sum_{1 \leq i, j \leq q} [\hat{Ric}(te_i, te_j)]^2 = \frac{1}{t^4} \|\hat{Ric}\|^2 + O\left(\frac{1}{t^2}\right) \\ (\|scal_t\|_t)^2 &= \frac{1}{t^4} \|\hat{scal}\|^2 + O\left(\frac{1}{t^2}\right) \end{aligned}$$

Therefore, at the point m we have:

$$(h_4)_t = \frac{1}{t^4} \hat{h}_4 + O\left(\frac{1}{t^2}\right) \quad (14)$$

This completes the proof of theorem B since the total space is compact. ■

Corollary 4.1 *1. The product $S^p \times M$ of an arbitrary compact manifold M with a sphere $S^p, p \geq 4$ admits a Riemannian metric with positive shwci.*

2. If a compact manifold admits a Riemannian foliation such that the leaves are with positive shwci then the manifold admits a Riemannian metric with positive shwci.

Proof. The first part is straightforward, to prove the second one it suffices to notice that the proof of the previous theorem works also in the case of local Riemannian submersions. ■

Corollary 4.2 *If a compact manifold M admits a free and smooth action of a compact connected Lie group G with rank r such that $r < [\frac{\dim G + 1}{2}]$ then the manifold M admits a Riemannian metric with positive shwci.*

Proof. The canonical projection $M \rightarrow M/G$ is in this case a smooth submersion. Let the fibers be equipped with a biinvariant metric from the group G via the canonical inclusion $\mathcal{G} \subset T_m M$.

Using any G -invariant metric on M , we define the horizontal distribution to which we lift up an arbitrary metric from the basis M/G . Thus we have defined a metric on M such that the projection $M \rightarrow M/G$ is a Riemannian submersion.

Finally, since the group G with a biinvariant metric is with positive shwci then so are the fibers with the induced metric, and we conclude using the previous theorem. ■

Remark. All simple Lie groups satisfy the property $r < [\frac{\dim G + 1}{2}]$.

5 Proof of Theorem C

We proceed as in Gromov-Lawson's proof for the case of scalar curvature [2].

Let (X, g) be a compact n -dimensional Riemannian manifold with positive scalar curvature and let $S^m \subset X$ be an embedded sphere of codimension q and with trivial normal bundle $N \equiv S^m \times \mathbf{R}^q$. There exists $r_0 > 0$ such that the exponential map $\exp : S^m \times D^q(r_0) \rightarrow X$ is an embedding, where for every $x \in S^m$, $\{x\} \times D^q(r_0)$ denotes the closed Euclidean ball in $\mathbf{R}^q \equiv \{x\} \times \mathbf{R}^q$. Let \exp^*g denotes the pull back of the metric g to the normal sub-bundle $S^m \times D^q(r_0)$.

Another natural metric on the normal bundle is the metric g^∇ defined using the normal connection ∇ , that is the metric compatible with the normal connection and such that the projection $S^m \times D^q(r) \rightarrow S^m$ is a Riemannian submersion. We shall denote also by g^∇ its restriction to the sub-bundles $S^m \times D^q(r)$ and $\partial(S^m \times D^q(r)) = S^m \times S^{q-1}(r)$.

Recall that at each $(p, v) \in S^m \times D^q(r)$ we have a natural g^∇ -orthogonal decomposition of the tangent space into vertical and horizontal subspaces, namely

$$T_{(p,v)}S^m \times D^q(r) = \mathcal{V}_{(p,v)} + \mathcal{H}_{(p,v)} \quad (15)$$

where $\mathcal{V}_{(p,v)}$ is the tangent space to the fiber (over p) $D^q(r)$ at v . These two metrics are tangent to the order two in the directions tangent to D^q , precisely with respect to the decomposition 15 we have (see [8])

$$\begin{pmatrix} g^\nabla + 0(r^2) & g^\nabla + 0(r) \\ g^\nabla + 0(r) & g^\nabla + 0(r) \end{pmatrix} \quad (16)$$

Remark. Note that in [2] in the begining of the proof of Lemma 2 page 430, it is claimed that the former metrics are sufficiently close in the C^2 -topology. But in general this is only true for the directions tangent to $S^{q-1}(r)$, a detailed study of the behavior of these two metrics will appear in a separate forthcoming paper [8]. The same error is also in [7]. A short proof of this fact is as follows:

With respect to the metric g^∇ , the sphere $S^m \hookrightarrow S^m \times D^q$ is totally geodesic (since for a Riemannian submersion the horizontal lift of a geodesic is a geodesic). But on the other side, the sphere $S^m \hookrightarrow S^m \times D^q$ is totally geodesic for the metric \exp^*g only if the sphere S^m is totally geodesic in (X, g) .

However this does not affect the corresponding conclusions in both papers (after minor changes) since the curvatures in question (that is the scalar

curvature and the p -curvatures, $p \leq q - 3$) of these two metrics on the bundles $S^m \times S^{q-1}(r)$ are high and close enough as $r \rightarrow 0$.

Now, it is easy to see that the second fundamental form of $S^m \times S^{q-1}(r)$ in $S^m \times D^q(r)$ with respect to the decomposition 15 is of the form

$$\begin{pmatrix} -\frac{\text{Id}}{r} & 0 \\ 0 & 0 \end{pmatrix} \quad (17)$$

Consequently, using formulas 16 and 17 one can deduce without difficulties that the second fundamental form of $S^m \times S^{q-1}(r)$ in $S^m \times D^q(r)$ with respect to the metric \exp^*g is of the form (with respect to the decomposition 15):

$$\begin{pmatrix} -\frac{\text{Id}}{r} + O(r) & O(1) \\ O(1) & O(1) \end{pmatrix} \quad (18)$$

Note that since the second fundamental form is a continuous function, then it still has the form 18 with respect to the following \exp^*g -orthogonal decomposition:

$$T_{(p,v)}S^m \times D^q(r) = \mathcal{V}_{(p,v)} \oplus \mathcal{H}'_{(p,v)} \quad (19)$$

where $\mathcal{V}_{(p,v)}$ is as in 15 and the distribution \mathcal{H}' is defined by the previous orthogonal decomposition. Note that as $r \rightarrow 0$, the distribution \mathcal{H}' converges to the distribution \mathcal{H} defined by the decomposition 15.

Now we define a hypersurface M in the product $S^m \times D^q(r_0)$ endowed with the product metric $\exp^*g \times \mathbf{R}$ by the relation

$$M = \{((x, v), t) \in S^m \times D^q(r_0) \times \mathbf{R} \mid (\|v\|, t) \in \gamma\}$$

where γ is a curve whose graph in the (r, t) -plane as pictured below:

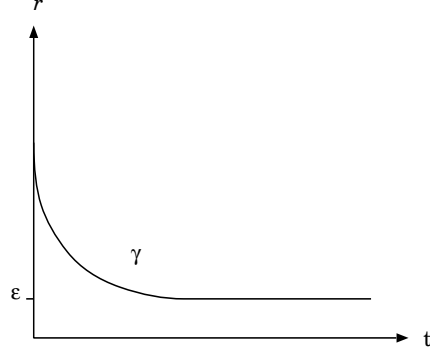
The important points about γ is that it is tangent to the r -axis at $t = 0$ and is constant for $r = \varepsilon > 0$. Thus the induced metric on M extends the metric \exp^*g on $S^m \times D^q(r_0)$ near its boundary and finishes with the product metric $(\partial(S^m \times D^q(\varepsilon)), \exp^*g) \times \mathbf{R} = (S^m \times S^{q-1}(\varepsilon), \exp^*g) \times \mathbf{R}$.

Next, we evaluate the *shwci* of the hypersurface M .

For each $m \in M$, we have the following \exp^*g -orthogonal decomposition of $T_m M$

$$T_m M = \mathbf{R}\tau \oplus \mathcal{V}_m \oplus \mathcal{H}'_m \quad (20)$$

where τ is the unit tangent vector to the curve γ in the (r, t) -plane and $\mathcal{V}_m, \mathcal{H}'_m$ are as in 19.



It results by a straightforward computation using 18 that the the second fundamental form of the hypersurface M has the following the form (with respect to the decomposition 20)

$$\begin{pmatrix} k & 0 & \dots & 0 \\ 0 & (-\frac{Id}{r} + O(r)) \sin \theta & (O(1) \sin \theta) \\ \vdots & (O(1) \sin \theta) & (O(1) \sin \theta) \\ 0 & & & \end{pmatrix} \quad (21)$$

where k denotes the curvature of the curve γ in the (r, t) -plane and θ denotes the angle between the normal to M and the t -axis at the corresponding point.

Then a long but direct computation using the Gauss equation and the previous formula 21 shows that the curvatures of M have the form :

$$\begin{aligned} \|R^M\|^2 &= \|R^{S^p \times D^q}\|^2 + \frac{(q-1)(q-2)}{2r^4} \sin^4 \theta + (q-1) \frac{k^2}{r^2} \sin^2 \theta + O(\frac{1}{r^2}) \sin \theta \\ \|\text{Ric}^M\|^2 &= \|\text{Ric}^{S^p \times D^q}\|^2 + \frac{(q-1)(q-2)^2}{r^4} \sin^4 \theta + q(q-1) \frac{k^2}{r^2} \sin^2 \theta \\ &\quad - \frac{(q-1)(q-2)^2 k}{r^3} \sin^3 \theta + O(\frac{1}{r^3}) \sin \theta \\ \|\text{scal}^M\|^2 &= \|\text{scal}^{S^p \times D^q}\|^2 + \frac{(q-1)^2(q-2)^2}{r^4} \sin^4 \theta + 4(q-1)^2 \frac{k^2}{r^2} \sin^2 \theta \\ &\quad - 2 \frac{(q-1)^2(q-2)k}{r^3} \sin^3 \theta + O(\frac{1}{r^3}) \sin \theta \end{aligned}$$

where we supposed that the curve γ has its curvature of the form $k = O(\frac{1}{r})$.

Consequently, we can evaluate the *shwci* of M as follows

$$h_4^M = h_4^{S^p \times D^q} + \frac{(q-1)(q-2)(q-3)(q-4)}{4r^4} \sin^4 \theta - \frac{(q-1)(q-2)(q-3)k}{2r^3} \sin^3 \theta + O\left(\frac{1}{r^3}\right) \sin \theta \quad (22)$$

Next we shall show that it is possible to choose the curve γ so that the metric induced on M has positive *shwci* at all points $m \in M$.

Formula 22 shows that for $\theta = 0$ we have $h_4^M = h_4^{S^p \times D^q}$ is positive, and then there exists an angle $\theta_0 > 0$ such that for all $0 < \theta \leq \theta_0$ the *shwci* of M is positive.

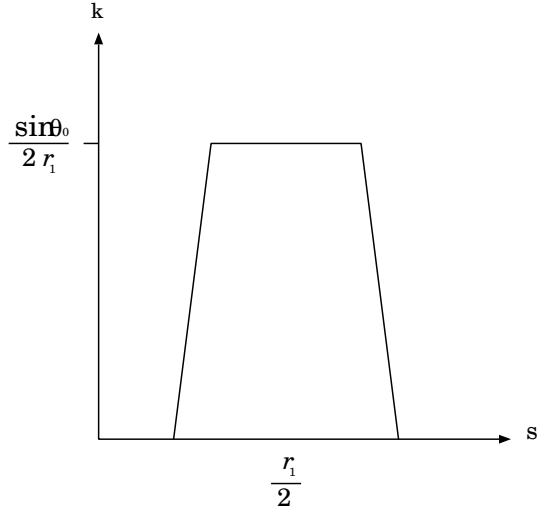
then we continue with a straight line ($k = 0$) of angle θ_0 , say γ_1 , until the term $\frac{(q-1)(q-2)(q-3)(q-4)}{4r^4} \sin^4 \theta_0$ is strongly dominating.

On the other hand, when $\theta = \pi/2$ then $k = 0$ and $r = \epsilon$ we have

$$h_4^M = \frac{(q-1)(q-2)(q-3)(q-4)}{4\epsilon^4} + O\left(\frac{1}{\epsilon^3}\right) \quad (23)$$

which is positive as ϵ is small enough and $q \geq 5$.

We now choose $r_1 > 0$ small and consider the point $(r_1, t_1) \in \gamma_1$. Then we bend the straight line γ_1 , beginning at this point, with a curvature $k(s)$ of the following form



where the variable s denotes the arc length along the curve.

Since $q \geq 5$, formula 22 shows that

$$h_4^M \geq h_4^{S^p \times D^q} + \frac{(q-1)(q-2)(q-3)}{2r^3} \sin^3 \theta \left(\frac{\sin \theta}{2r} - k \right) + O\left(\frac{1}{r^3}\right) \sin \theta \quad (24)$$

Then it is clear that the hypersurface M will continue to have $h_4^M > 0$, since $k < \frac{\sin \theta_0}{2r_1} < \frac{\sin \theta}{2r}$.

After this first bending, we have $\Delta r \leq \Delta s = \frac{r_1}{2}$ and then $r \geq r_1 - \Delta r \geq r_1 - \frac{r_1}{2} > 0$, consequently the curve will not cross the t -axis.

On the other hand, $\Delta \theta = \int k ds \approx \frac{\sin \theta_0}{4}$ is independent of r_1 . Clearly, by scaling down the curvature k , we can produce any $\Delta \theta$ such that $0 < \Delta \theta \leq \frac{\sin \theta_0}{4}$.

Our curve now continues with a new straight line γ_2 with angle $\theta_1 = \theta_0 + \Delta \theta$. By repeating this process finitely many times we can achieve a total bend of $\frac{\pi}{2}$.

Let g_ϵ denote the induced metric from $\exp^* g$ on $\partial(S^p \times D^q(\epsilon)) = S^p \times S^{q-1}(\epsilon)$, and recall that the new metric defined on M is the old metric when $t = 0$, and finishes with the product metric $g_\epsilon \times \mathbf{R}$. In the following we shall deform the product metric $g_\epsilon \times \mathbf{R}$ on $S^p \times S^{q-1}(\epsilon)$, to the standard product metric through metrics with positive *shwci*. This will be done in two steps:

Step1: We deform The metric g_ϵ on $S^m \times S^{q-1}(\epsilon)$ to the standard product metric $S^m(1) \times S^{q-1}(\epsilon)$ through metrics with positive *shwci*, as follows:

First, the metric g_ϵ can be homotoped through metrics with $h_4 > 0$ to the normal metric g^∇ since their *shwci* are respectively high and close enough, see formulas 23 and 14.

Then, for ϵ small enough, we can deform the normal metric g^∇ on $S^m \times S^{q-1}(\epsilon)$ through Riemannian submersions to a new metric where S^p is the standard sphere $S^p(1)$, keeping the horizontal distribution fixed.

This deformation keeps $h_4 > 0$ as far as ϵ is small enough, see formula 14.

Finally, we deform the horizontal distribution to the standard one and again by the same formula 14 this can be done keeping $h_4 > 0$.

Step2: Let us denote by $ds_t^2, 0 \leq t \leq 1$, the previous family of deformations on $S^m \times S^{q-1}(\epsilon)$. They are all with positive *shwci*. Where $ds_0 = g_\epsilon$ and ds_1 is the standard product metric.

It is clear that the metric $ds_t^2 + dt^2, 0 \leq t \leq a$, glues together the two metrics $ds_0 \times \mathbf{R}$ and $ds_1 \times \mathbf{R}$. Furthermore, there exists $a_0 > 0$ such that for all $a \geq a_0$ the metric $ds_t^2 + dt^2$ on $S^m \times S^{q-1}(\epsilon) \times [0, a]$ is with positive *shwci*. In fact, via a change of variable, this is equivalent to the existence of $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, the metric $\lambda^2 ds_t^2 + dt^2$ is with positive *shwci*. This is already known to be true again by formula 14. ■

Corollary 5.1 *Let G be a finitely presented group. Then for every $n \geq 6$, there exists a compact n -manifold M with positive shwci such that $\pi_1(M) = G$.*

Proof. Let G be a group which has a presentation consisting of k generators x_1, x_2, \dots, x_k and l relations r_1, r_2, \dots, r_l .

Let $S^1 \times S^{n-1}$ be endowed with the standard product metric which is with positive shwci ($n - 1 \geq 4$). Remark that the fundamental group of $S^1 \times S^{n-1}$ is infinite cyclic. Hence by taking the connected sum N of k -copies of $S^1 \times S^{n-1}$ we obtain an orientable compact n -manifold with positive shwci (since this operation is a surgery of codimension $n \geq 5$). By Van-Kampen theorem, the fundamental group of N is a free group on n -generators, which we may denote by x_1, x_2, \dots, x_k .

We now perform surgery l -times on the manifold N , killing in succession the elements r_1, r_2, \dots, r_l . The result will be a compact, orientable n -manifold M with positive shwci (since the surgery is of codimension $n - 1 \geq 5$) such that $\pi_1(M) = G$, as required. ■

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